# An analytical study of sums of two squares and powers of positive integers in relation to the Hasse Principle

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**ABSTRACT**

In this research paper aims to study the sum of Two squares and powers of positive integers $k$ which cannot be written in the mathematical form $u^2 + v^2 + z^k = n$ for all values of $n \geq 3$ for the positive integers u, v, z. It also aims to study a number of example that are consistent with its principle or are considered counterexamples to Hasse Principle, as well as the strong approximation of the results for positive integers resulting from the aforementioned mathematical formula. And also by linking the existence of solutions to the Diophantine equations on rational numbers $\mathbb{Q}$, And also solutions for complements of rational numbers $\mathbb{Q}_p$.

**KEYWORDS:** Diophantine equations, approximation of principle, waring type problem, Strong approximation, Ternary additive problems.

**الملخص.**

يهدف هذا البحث إلى دراسة مجموع مربعين وقوى للأعداد الصحيحة الموجبة $k$ والتي لايمكن كتابتها بالصيغة الرياضية $u^2 + v^2 + z^k = n$ لجميع قيم $n \geq 3$ للأعداد الصحيحة الموجبة u, v, z. كما يهدف هذا البحث إلى دراسة عدد من الأمثلة التي تتوافق مع مبدأ هاس أو التي تعتبر أمثلة مضادة له، بالإضافة إلى التقريب القوي لنتائج الأعداد الصحيحة الموجبة الناتجة عن الصيغة الرياضية المذكورة أعلاه. وكذلك أيضاً من خلال ربط وجود حلول لمعادلات ديوفانتين على الأعداد النسبية $\mathbb{Q}$ وكذلك حلول لمكملات الأعذاد النسبية $\mathbb{Q}_p$.

## 1. INTRODUCTION

Hasse's Principle: If the equation $u^2 + v^2 + z^k = n$ has solutions in all completions of $\mathbb{Q}$ (i.e, $\mathbb{R}$ *and* $\mathbb{Q}_p$ for all primes $p$) does it necessarily have a solution in $\mathbb{Q}$ (and thus in $\mathbb{Z}$)?. The Hasse Principle holds for quadratic forms, but often fails for higher-degree equations. The paper likely identifies cases where the equation has solutions in $\mathbb{R}$ *and* $\mathbb{Q}_p$ for all $p$ but no solution in $\mathbb{Z}$, violating the Hasse principle.

For fixed $k \geq 1$ let $S_k(N)$ be the set of positive integer's $n$, with

$$3 \leq n \leq N$$

$$n \not\equiv 0 \bmod 2, \qquad n \not\equiv 2 \bmod 2, \qquad for\ odd\ k$$

$n \equiv 3 \; mod \; 24, \qquad\qquad for \; even \; k$

$n \not\equiv 0 \; mod \; 5, \qquad\qquad for \; k \equiv 2 \; mod 4$

$n \not\equiv 0 \; mod \; 5, \qquad\qquad for \; k \equiv 0 \; mod \; 4$

$n \not\equiv 0.2 \; mod \; 5, \qquad\qquad for \; k \equiv 0 \; mod \; 4$

$n \not\equiv 1 \; mod \; p, \qquad for \; each \; p \equiv 3 \; mod \; 4 \; with \; (\frac{p-1}{k})$

Then the number of integers $n \epsilon S_k(N)$ not of the form

$n = p_1^2 + p_2^2 + p_3^k$

Is for all $B > 0$, at most

$O_B(\frac{N}{(logN)^B})$

This enhanced a Hua result. [11,Theorem1], Who demonstrated this with $B = \frac{k}{k+2}$.

As we have previously investigated solutions for $u^2 + v^2 + z^k = n$, Because of this relationship, we have decided to contribute without restricting the variables to only primes, which should lead to stronger results. This work builds upon foundational studies initiated by Davenport and Heilbronn [3] and expanded upon by numerous other writers, including Brudern [5] has illustrated that there are at most $O(N^{1-\frac{1}{k}+\epsilon})$ integers $n \leq N$ with no solutions of

$$u^2 + v^2 + z^k = n \qquad (1.1)$$

where $n$ is not belong to a residue class that congruence obstacles exclude. See also for a summary of findings regarding sums of mixed powers. [9] and [14].

For every sufficiently big $n$, it was usually assumed that the Hasse principle for Eq.(1.1) would guarantee a solution, for any such $n$ instance meeting the necessary congruence conditions (1.1) As outlined in Chap.8 in [15], this suggested that, in positive integers, Eq.(1.1) always had a solution under these conditions. However, Jagy and Kaplansky[2] disproved this in 1995 by demonstrating that for $k = 9$ and there are at least a few positive constants $C \frac{N^{\frac{1}{3}}}{logN}$ integers that are positive $n \leq N$ that don't add up to one kth power and two squares.

Furthermore, their approach is effective for any odd composite value of k, but does not extend for other k values. In [10] we demonstrated that a comparable constraint applies to that method for $k = 4$. which does in fact generalize to all multiples of four (see Theorem 3.1). By modifying their approach further, we can not only enlarge the set of exceptional $n$ but we can also address cases where $k$ is not divisible by four. Specifically, we establish that equation (1.1) does not satisfy 'strong approximation': For $k \equiv 2$ (mod 4), $k \geq 6$ and sufficiently large $N$ we demonstrate that there are asymptotically $\gg \frac{N^{\frac{1}{2}}}{(\log N)^{\frac{1}{2}}}$ positive integers $n \leq N$ for which Eq. (1.1) lacks solutions with $z$ constrained to a particular residue class, despite the absence of any congruence obstructions (see Theorem 3.2). For odd $\geq 3$, we further establish that there are at least $\frac{kN^{\frac{1}{2}}}{2\varphi(k)\log N}$ such remarkable positive integers $n \leq N$ (see Theorem 2.1).

Additionally, it is worth noting that Hooley [3] investigated sums of a $k$th power and three squares, Friedlander and Wooley [1] sums of three biquadrates and two squares, and Wooley [13] sums of squares and a 'micro square', in relation to a conjecture of Linnik. In this context, we would like to highlight an apparently overlooked historical reference: Rieger's Theorem (7) [4], affirms that the quantity of integers $n \leq N$ which may be expressed as $u^2 + v^2 + z^k = n$, where $z \leq F(N)$, and $F$ is a function that has a monotonic tendency to infinity, with $F(n) \leq \sqrt{\log N}$ , is $\gg_{k,F} \frac{N\,F(N)}{\sqrt{\log N}}$ , in other words, as good as it can be.

## 2. TWO SQUARES AND AN ODD KTH POWER.

**Theorem 2.1** let $k \geq 3$ be odd. If p is a prime number with $p \equiv 1$mod 4k. Thus, there aren't any integers $u, v, z$ positive or negative, with $u^2 + v^2 + z^k = p^k \; and \; z \equiv 2k \; mod \; 4k$.

**Proof** Assume the equation admits solutions, then $u^2 + v^2 = (p - z)(p^{k-1} + p^{k-2}z + \cdots + pz^{k-2} + z^{k-1})$.

If $z \equiv 2k \; mod \; 4k$, then $p - z \equiv 2k + 1 \; mod \; 4k$. Since $k$ is odd, $2k + 1 \equiv 3 \; mod \; 4$.

Hence $p - z$ must contain a prime divisor $q \equiv 3 \; mod \; 4$ with odd multiplicity. Note that $\gcd(q, k) = 1$, as otherwise ${}^{q}/_{k}$ and $0 \equiv p - z \equiv 2k + 1 \equiv 1 \; mod \; q$ demonstrates a contradiction.

Remember that squares the integer according to the standard classification of integers, which can be written as the sums of two squares, $u^2 + v^2$, includes the primary factors $q \equiv 3 \; mod \; 4$ with even multiplicity only. Therefore both $p - z \; and \; (p^{k-1} + p^{k-2}z + \cdots + pz^{k-2} + z^{k-1})$ are divisible by $q$. With $p \equiv z \; mod \; q$ it follows that

$$(p^{k-1} + p^{k-2}z + \cdots + pz^{k-2} + z^{k-1}) \equiv kz^{k-1} \equiv 0 \; mod \; q$$

This implies that $\frac{q}{z} \; and \; hence \; \frac{q}{p}$, which is impossible, as $q = p$ would contradict $p \equiv 1 \; mod \; 4$.

Furthermore given there are no congruence barriers that would suggest that in $u^2 + v^2 + z^k = p^k$ it follows that there are no solutions when $z \equiv 2k \; mod \; 4k$.

First, consider selecting an integer for a fixed odd prime q in order to identify this. $z \equiv 2k \; mod \; 4k$ such that $q$ is coprime to $p^k - z^k$; similarly, for $q \equiv 8$ just choose $z \equiv 2k$. Here, the congruence for this fixed z, $u^2 + v^2 + z^k = p^k \; mod \; q$ has a nonsingular solution in $u$ and $u$ which by Hensel's lemma can be lifted to a $q \; adic \; or \; 2 \; adic$ solution respectively.

According to arithmetic progressions' prime number theory, the quantity of these instances, $p^k \leq N$ with $p \equiv 1 \; mod \; 4k$, is asymptotically.

$$\frac{1}{\varphi(4k)} \int_2^{N^{\frac{1}{k}}} \frac{dt}{\log t} \sim \frac{k}{2\varphi(k)} \frac{N^{\frac{1}{k}}}{\log N}$$

## 3. TWO SQUARES AND AN EVEN KTH POWER

### 3.1 A kth power and two squares, $\mathbf{k \equiv 0 \; mod \; 4}$.

**Theorem 3.1** Suppose that $\frac{4}{k}$ and let $p$ be prime with $p \equiv 7 \; mod \; 8$. Let $n \equiv 1 \; mod \; 8$ be 1 or consist solely of prime components that are equivalent to 1 mod 4, and presume that $n < p$. Then, no positive integers exist $u, v, z \; with \; u^2 + v^2 + z^2 = (np)^2$.

**Proof** Let $k = 2t$, where t is even. If there are answers, then $u^2 + v^2 = (np - z^t)(np + z^t)$. If $z$ is even, then $np - z^t \equiv 3 \; mod \; 4$. If $z$ is odd, then $np - z^t \equiv 6 \; mod \; 8$. In both situations $np - z^t$ has to have a prime divisor $q \equiv 3 \; mod \; 4$ has an odd multiplicity.

Thus, similar to the theorem (2.1) proof, we deduce that both $np - z^t$ and $np + z^t$ are divisible by $q$. Since $2n \not\equiv 0 \; mod \; q$, and since $p$ is prime: $p = q$, and since $z \neq 0: q$ divides $z$. But this gives a contradiction:

$$u^2 + v^2 + z^k > q^k \geq q^4 > (np)^2 = (np)^2$$

An estimation of the number of integers $np \leq N$, with $n \equiv 1 \; mod \; 8$ consisting of prime factors 1 mod 4 only is of order of magnitude $\frac{N}{(\log N)^{\frac{1}{2}}}$, and roughly 50% of these figures meet the congruence requirement $n \equiv 1 \; mod \; 8$.

Let $f: \mathbb{N} \longrightarrow \{0,1\}$ be the characteristic function of these integers $n$, i.e. we put $f(n) = 1, \; if$

$n \equiv 1 \; mod \; 8$, and all prime factors of $n \; are \; 1 \; mod \; 4$; otherwise, we put $f(n) = 0$. Now

$$\sum_{np \leq N, n < p} f(n) = \sum_{n \leq \frac{n}{p}, n < p} f(n) \gg \sum_{N^{\frac{1}{2}} \leq p \leq N^{\frac{3}{4}}} \frac{\frac{N}{p}}{(\log\left(\frac{N}{p}\right))^{\frac{1}{2}}} \gg \frac{N}{(\log N)^{\frac{1}{2}}}$$

Where we used that

$$\sum_{N^{\frac{1}{2}} \leq p \leq N^{\frac{3}{4}}} \frac{1}{p} = loglogN^{\frac{3}{4}} - loglogN^{\frac{1}{2}} + o(1) = \log\left(\frac{3}{2}\right) + o(1) \gg 1$$

This order is the correct order of magnitude in light of Landau's theorem. Thus, the quantity of extraordinary $(np)^2 \leq N$ provided by Theorem (3.1) is $\gg \frac{N^{\frac{1}{2}}}{(\log N)^{\frac{1}{2}}}$.

Keep in mind that with regard to Theorem (2.1), one can verify that there are no obstacles to congruence for the representation of $(np)^2$.

## 3.2 A kth power and two squares, $k \equiv 2 \bmod 4$

**Theorem 3.2** Suppose that $k \equiv 2\, mod\, 4$, $k \geq 6$ and let $p$ be a prime with $p \equiv 7\, mod\, 8$. Let $n < p$ be an integer that is either 1 or made up of prime elements that are only congruent to 1 mod 4, and $n \equiv 1\, mod\, 8$. Then there are no positive integers $u, v, z\, where\, \frac{2}{z}\, with\, u^2 + v^2 + z^2 = (np)^2$.

**Proof:**

The proof is nearly exactly as stated above.

From the previous example, it is evident that the number of exceptional $(np)^2 \leq N$ given by Theorem3.2 is $\gg \frac{N^{\frac{1}{2}}}{(\log N)^{\frac{1}{2}}}$. Furthermore, note that similarly to the Theorem 2.1 It is noted that the required representation of $(np)^2$does not encounter any congruence obstacles.

## 4. CONCLUSION

The primary conclusion of this study is that while the Hasse principle holds for some quadratic equations, it often fails in higher-degree cases such as Fermat-type equations. The failure of the principle illustrates the need for additional tools, such as Selmer groups and the Brauer-Manin obstruction, to fully characterize solvability in rational numbers. This analysis underscores the deep connections between number theory, Diophantine equations, and modern algebraic techniques. These findings not only contribute counterexamples and refinements to classical results but also strengthen the conceptual bridge between the arithmetic of sums of squares, higher powers, and the broader context of rational solutions.

Further research in this area may continue to reveal unexpected phenomena at the intersection of analytic number theory and the Hasse Principle.

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